THE

MATHEMATICAL GAZETTE.

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WITH THE CO-OPERATION OF

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THE TEACHING OF MATHEMATICS.

THE British Association this year has inaugurated a new Section, that of Education; and at the recent Glasgow meeting this new Section made a good start, under the presidency of Sir John Gorst, by holding a series of joint discussions with several of the older Sections, in addition to the usual reading of sectional papers. Thus it came that on Saturday the 14th of September a three hours' discussion took place on the "Teaching of Mathematics."

Whatever may be thought of the results of the discussion, there could be no question as to the sustained interest of the debate both on the part of the speakers and on that of the listeners, who must have numbered about two hundred.

Professor John Perry, F.R.S., opened by reading extracts from an elaborate paper which he had prepared, printed copies of which were circulated in the room. This paper (which, along with abstracts of the discussion, Professor Perry has promised to publish) contained the case for reform in a series of pregnant paragraphs, followed by syllabuses of elementary and advanced courses of mathematics, designed to show the lines on which mathematical teaching ought to run. It was only by restricting the time of each debater to ten minutes that the Chairman was able to call on all those who had in advance promised to take part in the discussion; and one could feel that a large proportion of the silent members were so perforce, and would dearly have liked to unburthen their minds of those thoughts and feelings that were stirred up in the course of the debate.

Writing a week after the meeting, with only an imperfect newspaper report to aid my memory, I can only give a personal reminiscence rather than a summary of the debate.

Professor Perry's name is now pretty well identified with the movement towards making the teaching of mathematics more practical, especially with a view to the needs of the engineer. His standpoint may be gathered from the following extracts from his paper:

"It is usefulness which must determine what subjects ought to be taught to children, and in what ways." . . .

"All advocates of orthodox methods . . . seem willing to sacrifice every form of usefulness to one form, the emotional or soul-preserving mind-training inherent in a perfect logical system; a huge complex deduced logically from simple fundamental truths." . . .

"Where would be the harm in letting a boy assume the truth of many propositions in the first four books of Euclid, letting him accept their truth partly by faith, partly by trial? Giving him the whole fifth book of Euclid by simple algebra? Letting him assume the sixth book to be axiomatic? Letting him, in fact, begin his severer studies where he is now in the habit of leaving off?" . .

"I feel that throughout one's whole mathematical course it is important to teach a student through his own experiments, through concrete examples worked out by him. Even good mathematical teachers hate to see their students 'wasting time,' as they call it, in actually plotting lines of force or stream lines after the algebraic academic answer has been arrived at. They would probably call it waste of time to caulk the joints of plates in a ship. Without this kind of illustration I feel sure that the whole study is useless except for one man in a thousand. I am here speaking of advanced work. But even in elementary work a student ought to be induced to apply his mathematics to problems in his own experience."

It may be added that Professor Perry's strictures on the curricula of the Continental polytechnics were even more severe than those he applied to the mathematical courses of our schools in Britain.

Among those who contributed to the discussion were six Professors of Mathematics, three Professors of Physics, one of Engineering, and one of Natural History. The President of Section A (Mathematics), a member of Parliament, and several school teachers, including two ladies, also spoke.

With some of Professor Perry's contentions there was general agreement. All agreed that the elementary teaching ought to be less pedantic and more concrete, a lady pointing out that at present the young scholar has to overcome two difficulties simultaneously, that of becoming acquainted with mathematical conceptions, and that of a first introduction to formal deductive reasoning—difficulties that ought to be taken separately, by giving the child a course of practical geometrical drawing, and the handling of simple solids, before Euclid is taken up. Professor Sylvanus Thompson insisted on the importance of adding Drawing as a fourth subject to stand on the same footing as the three R's.

There seemed to be a general assent, even on the part of the "official" mathematicians, to the idea of "abandoning Euclid"—a notable fact; and to Major Macmahon's query, "Who, then, are we to put in Euclid's place?" the answer was given by Professor Louis Miall, who pointed out that on the Continent Euclid has been abandoned without any authority having been put in his place.

Exception was taken by Professor Forsyth, of Cambridge, to the opener's idea that it is possible to put the beginner quickly in possession of all the results of existing mathematics, to enable him to extend the boundaries of knowledge. He, however, was in agreement with the doctrine that the young student is not ripe for the philosophy of mathematics seeing that the greatest mathematicians are not yet agreed upon fundamental principles.

One of the ladies pointed out that Professor Perry's syllabus seemed to have been drawn up for boys who were to be engineers, and asked what was to be done in schools for girls who also wanted a mathematical course. Professor Everett, too, was inclined to think the syllabus suited to the needs rather of a technical than of a liberal education.

During the first half of the discussion the question as to where the responsibility rests for the present unsatisfactory state of affairs was scarcely touched, but the President of the Association (Professor Rücker) set that ball rolling by remarking that the present examination system was one imposed on the teachers from the outside, as a test of their efficiency, and that the teachers were not really to blame for it. This suggestion was pursued by Professor Thompson, who cited an instance in which a reform in the teaching of arithmetic in a certain school was stamped out by the inspector, and

asserted that the same kind of strangling of initiative was occurring daily. Professor Louis Miall, in an interesting and informing speech, which was heartily applauded, pointed out that the true method of examination was that in which the teacher himself examined, with an outside examiner present to see fair play, as it were. This proposition was perhaps the best answer to the rather malicious question put by one of the lady members, "Who is to teach the teachers and examine the examiners?"

The discussion was closed by a short speech from the sectional President, Sir John Gorst, who gave some reminiscences of his early experiences in teaching mathematics to the Maories, and expressed his conviction of the value of such a discussion in which the teachers and the specialists are brought into direct contact.

I think there can be no doubt of the benefit in ventilating what is really a burning educational question, though, of course, it is difficult for such an assembly to get at the "hard pan" of the crucial questions in one sitting. May I hazard the opinion that the line of reform indicated by Professor Miall, that of emancipating the teacher from all detailed syllabuses, is a more hopeful one than that of merely laying down a fresh syllabus, which in the nature of things will in course of time be as much of a clog upon educational progress as that which it replaces? R. F. MUIRHEAD.

THE SLIDE RULE.

Having accepted with some mental reservation the assurance of the Editor of this Gazette that the Slide Rule, although invented by Gunter as far back as 1624, is still unknown to many mathematicians, the writer has ventured to make this article explanatory and descriptive. To such then as are unenlightened he offers the good news that half-a-guinea will purchase an instrument capable of multiplying and dividing, of squaring and cubing, of extracting square and cube roots, of solving triangles and performing rapidly many similar combinations of arithmetic and trigonometric operations, all without any mental effort and to an accuracy of from $\frac{1}{5}$ to $\frac{1}{3}$ %. To examiners having to reduce marks to percentages the Slide Rule is invaluable, to physicists it is indispensable, to engineers it is both invaluable and indispensable.

The Gravet Slide Rule has been selected for description, mainly because Gravet has for over thirty years set the fashion in the manufacture of accurate yet inexpensive instruments. The detailed explanation has been restricted to the more elementary operations, for when the principles of these have been grasped no mathematician can find difficulty in extending them.

The operations of the Slide Rule may best be illustrated by studying the effect of sliding one ordinary uniformly graduated scale along another such scale. Thus in Fig. 1 the scale B has been slid along the scale A until its extreme left-hand graduation, marked O, comes under the graduation 17 on A: we note now that each graduation on B marks on A the result of adding 17 to it; thus above the graduation 5 on B we find 22 (*i.e.* $5+17$) on A, above 32 on B we find 49 (*i.e.* $32+17$) on A. Hence "the addition $(x+y)$ may be performed by setting the left end of B to x on A and reading the sum on A at the point marked by y on B." Conversely, the subtraction $(x-y)$ may be performed by moving the slide B till the graduation y on B comes under x on A, the left end of B then marks $(x-y)$ on A. In Fig. 1 the operation $44-27=17$ is indicated.

If, instead of making the graduations on A and B identical, we replace B by a scale C marked off to twice as big a scale as A, we have a ready method of halving or doubling numbers. For when A and C are placed "zero to

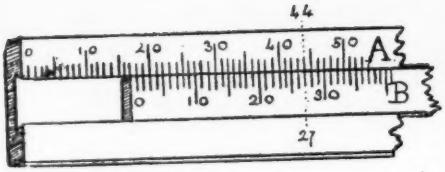

Fig. 1

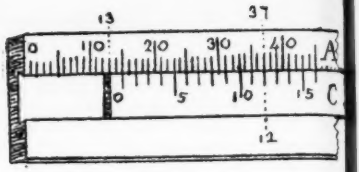

Fig. 2

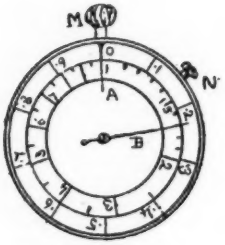

Fig. 8

Fig 9

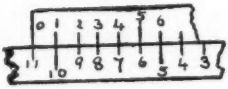

Fig 11

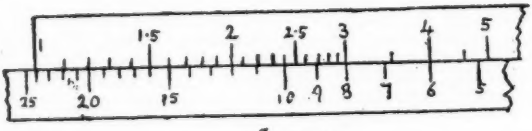

Fig 12.

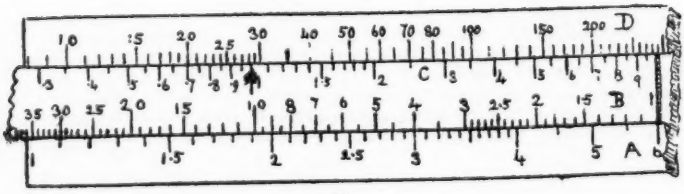

Fig. 13

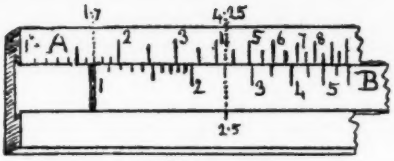

Fig 4

Fig 5

Fig 3

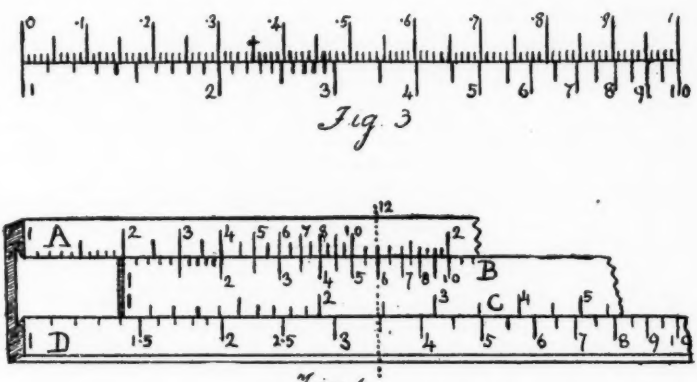

Fig 6

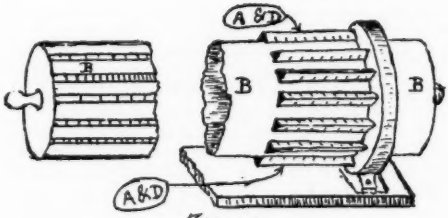

Fig 10

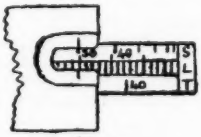

Fig 7

zero" we find at any point on C half of the number marked on A immediately above it. Fig. 2 shows how such a rule performs the operation $(x+2y)$. The left end of C is set to x on A, the sum $(x+2y)$ is then read on A at the point marked by y on C; the figure gives

$$13+2\times10=33, \quad 13+2\times12=37, \text{ and so on.}$$

Similarly by interchanging the functions of A and C we may perform the operation $(x+\frac{1}{2}y)$; thus, set the left end of A to mark 8 on C, then $8+\frac{1}{2}\times12=14$ is read on C at the point marked by 12 on A.

By constructing a rule in which the scale of C bears any other relation to that of A, say k times as big, we can in the same manner perform the operations $(x+ky)$ and $\left(x+\frac{y}{k}\right)$.

The essential feature of the above processes is that by sliding one scale along another we can *mechanically add two lengths*, and these lengths being proportional to numbers, it follows that we at the same time add two numbers. If now instead of making the lengths proportional to numbers we make them proportional to the *logarithms* of numbers, we shall by the process of Fig. 1 mechanically add $\log x$ to $\log y$ and thus obtain a length proportional to $\log(xy)$, and by the converse subtraction-process we shall take $\log y$ from $\log x$ and thus obtain a length proportional to $\log\left(\frac{x}{y}\right)$.

The construction of logarithmic scales is quite simple; Fig. 3 shows the method. Above the line is a uniform scale, below it the logarithmic scale. Since $\log 1=0$, the graduation 1 is placed at the left end of the scale; since $\log 2=\cdot301$, the graduation 2 is placed slightly beyond ·3 of the upper scale and so on.

$\log 1 = \cdot0$
$\log 1\cdot5 = \cdot176$
$\log 2 = \cdot301$
$\log 3 = \cdot477$
$\log 4 = \cdot602$
$\log 5 = \cdot699$
$\log 6 = \cdot778$

It is interesting to note how the graduations close together towards the right end of the scale, the intervals between them representing $\delta . \log x$ being approximately proportional to $\frac{1}{x}$. In the figure only the main graduations 1, 1·1, 1·2, etc., 3, 3·5, 4, 4·5, etc., are shown, but on the ordinary 10-inch slide rule, subdivisions are marked at 1·02, 1·04, and so on from 1 to 2; 2·05, 2·1, 2·15, etc., from 2 to 5; 5·1, 5·2, 5·3, etc., from 5 to 10; further subdivision is made by the eye, and very little practice is needed to attain an accuracy of 1 in 500. It might at first sight seem that there is a loss of accuracy at the right end of the scale, but this is only apparent, the percentage error being the same throughout.

Since $\log 20=1\cdot301$, $\log 30=1\cdot477$, and so on, it follows that the extension of the scale beyond 10 up to 100 involves merely the repetition of the scale from 1 to 10. So a further extension from 100 to 1000 involves another repetition of the same scale. But just as a single four-figure table of logarithms of numbers from 1000 to 9999 serves to give us the logarithms of say 2538, 25·38, 2·538, provided due regard be paid to the characteristic, so a single scale from 1 to 10 serves for all numbers provided a watchful eye be kept on the decimal point in the result. In practice it is convenient to have the scale repeated once.

Fig. 4 shows the logarithmic scale B slid along a similar scale A till its left end marks 1·7 thereon; the arrangement is analogous to that of Fig. 1, but here we add *log* 1·7 to a series of *logs* of numbers. Hence above 2 on B we find 3·4 (*i.e.* $1\cdot7\times2$) on A, above 3 on B we find 5·1 (*i.e.* $1\cdot7\times3$) on A and so on. In general the operation $x\times y$ may be performed by setting the left end of B to x upon A, and reading the product on A at the point marked by y on B.

Fig. 4 also shows the process of division which is analogous to the subtraction shown in Fig. 1. Thus the division $\frac{x}{y}$ $\left(e.g.\ \frac{4\cdot25}{2\cdot5}\right)$ may be performed

by moving the slide B until the graduation y (2·5) upon it comes under x (4·25) on A, the left end of B then marks $\frac{x}{y}$ (1·7) on A.

Now let us construct on the lower edge of the slide a logarithmic scale C twice as long as A or B, and also on the lower edge of the rule construct D identical with C. These scales C and D may be used as a pair like A and B to multiply and divide; but their main use is in combination with A or B.

Note that the pair of scales B and C [and also the pair A and D] are fixed relatively to each other in the position "unity to unity," and that this position corresponds to the position "zero to zero" of the scales A and C in Fig. 2. They, therefore, give automatically the halves and the doubles of logarithms, *i.e.* they "square" or they extract the square root. As the scales A and D are not adjacent, a sliding index or "cursor" is needed (Fig. 5); it consists of a light metal frame carrying a small plate of glass with a fine perpendicular line engraved on its under surface.

Set the index to mark any number x on A (or B), then it also marks $\sqrt{x}$ on D (or C). In Fig. 6 the index stands at 12 on A, 3·462 ($=\sqrt{12}$) is marked on D; it stands at 6 on B, 2·448 ($=\sqrt{6}$) is marked on C.

Further, Fig. 6 shows how the rule performs the following operations:

(*a*) $\sqrt{x}\times y$: the left end of B is set to x on A, the product $\sqrt{x}\times y$ is then read on D at the point marked by y on C; thus

$$\sqrt{2}\times 2=2{\cdot}83,\ \sqrt{2}\times 3{\cdot}2=4{\cdot}525,\text{ and so on.}$$

(*b*) $\frac{\sqrt{x}}{y}$: the index is set to x (12) on A, y (2·45) on C is then pushed under the index, and the quotient 1·414 is marked on D by the left end of C.

(*c*) $x\times y^2$: the slide is set so that the left end of C marks y (=1·414) on D, the product xy^2 ($3\times 1{\cdot}414^2=6$) is then read on A at the point marked by x (3) on B.

(*d*) $\frac{x}{y^2}$: the index is set to x (12) on A, y (2·45) on C is then pushed under the index, the quotient $\frac{x}{y^2}$ (2) is marked on A by the left end of B. Note that this operation only differs from (*b*) in that the result is read on A instead of C; this is, of course, because $\frac{\sqrt{x}}{y}=\sqrt{\frac{x}{y^2}}$.

The sliding index or cursor was added by M. Gravet of Paris about 1882, and added immensely to the utility of the rule, for it enables us to obtain the final result of several multiplications and divisions without recording the intermediate steps.

For example, to evaluate $\frac{42{\cdot}5\times 362}{2{\cdot}5\times 11{\cdot}8}$.

(*a*) Set the index to 425 on A.

(*b*) Push the slide till 2·5 on B comes under index as in Fig. 4.

[The left end of B now marks the quotient 170 on A; but there is *no need to read off this quotient*, for the first step towards the next operation—multiplication by 362—is to set the left end of B to 170, *i.e.* to the point at which it is now standing.]

(*c*) Set the index to 362 on B.

[This marks the product 617 on A, but this again being an intermediate result need not be read.]

(*d*) Push the slide till 11·8 on B comes under the index, then the final result 522 is read on A.

Rules may be formulated to decide where the decimal place should stand in the result, but the intelligent computator will know *a priori*, say, whether a mass he is dealing with is 522, 52·2, or 0·522 grammes.

By using the scales *A*, *B*, *C*, *D* in judicious combination operations involving any number of multiplications or divisions by simple numbers, by their squares or by their roots may be performed.

The process of extracting a cube root is interesting, it is analogous to finding one-third of a number with the rule shown in Fig. 2. Here if we set an index at 39, and adjust the slide until the number on *A* marked by the left end of *B* is identical with the number marked on *B* by the index, this number is obviously $\frac{1}{3}$ of 39, to wit 13.

So with the rule of Fig. 6: to find $\sqrt[3]{8}$; set the index at 8 on *A*, adjust the slide until the number on *A* marked by the left (or right) end of *B* is identical with the number marked on *C* by the index: the log of this number is obviously $\frac{1}{3}$ of log 8, hence the number is $\sqrt[3]{8}$, to wit 2.

Having set the index to 8 on *A* the above adjustment can be made in three and only three possible ways, one gives $\sqrt[3]{8}$, the others $\sqrt[3]{80}$ and $\sqrt[3]{800}$; just as the logarithms ·90309 with its different characteristics 0, 1, 2, 3, 4 gives three distinct mantissae after division by 3.

In practice it is necessary to decide on the first figure of the result by inspection before beginning to adjust the slide; *e.g.* $\sqrt[3]{\cdot 0857}$, the first figure is 4.

The under surface of the slide has three scales engraved upon it. The middle one is uniformly divided, it is in fact the uniform scale from which *D* has been constructed (as in Fig. 3). This scale gives by means of a fixed index mark on a slot at the back of the rule the logarithm of the number to which the slide has been set on *D*. The upper scale *S* is constructed by marking off lengths proportional to log sines of angles to the same scale as *A* and *B*; it readily gives sines, cosines, secants, cosecants, and their logs. The lower scale *T* is constructed from log tangents; in the French-made rules it also is to the same scale as *A* and *B*; on most German and so-called English rules, it is to the same scale as *C* and *D*; it gives tangents, cotangents, and their logs by means of a slot at the left end of the rule. [Some makers omit this slot—a very serious drawback.]

These three scales may also be used by taking the slide out of the rule and replacing it bottom face uppermost. This is convenient when several sines or tangents have to be multiplied by a constant as in reducing galvanometer readings; for if one end of the slide be set to the constant on *A* (possibly on *D* for tangents) the products for each angle are read off on *A* where marked by the graduation on *S* or *T*.

There is a distinct type of Slide Rule in which a circular logarithmic scale is constructed on a disc capable of rotation (Fig. 8).

Here the mechanical addition of logarithms is effected by means of two pointers; one *A* is fixed, the other *B* can be turned about the centre like a watch hand. If the disc be set so that "1" comes under the fixed index *A*, and the index *B* be then set to any number (*e.g.* 1·7), the angular distance *A* to *B* then represents log 1·7; if now the disc be turned to a new position, say, so that 22 comes under *A*, *B* remains at the same distance as before ahead of *A*, hence it adds log 1·7 to log 22, and we therefore find the product 37·4 (=1·7 × 22) marked by *B* on the disc. Division is equally simple.

Probably the best rule of this type is "Boucher's Calculator," it is of the size of a large watch, it has on one face in concentric circles a scale for sines, a scale for numbers, a scale of numbers twice the size occupying two complete circles and corresponding to the *D* scale of the straight rule—this scale gives square roots; on the other face is a scale of numbers enlarged three times—it therefore occupies three complete circles and gives cube roots, on this face also is a scale of log sines and a uniform scale giving logarithms. Two milled heads *M* and *N* turn the disc and the index respectively.

The advantage of these rules is that they are very compact and handy, and are beautifully engraved; the disadvantage is that they cost 27s. 6d., and

that there is great liability to parallax error in reading—for the movable index in turning has to clear the fixed index and therefore must be comparatively far from the disc; this cause limits the accuracy of calculation to about $\frac{1}{2}$ % or 1 %.

Abundant provision has been made for those computers for whom an accuracy of $\frac{1}{5}$ % does not suffice. Thus straight rules of the Gravet pattern are made 20 inches long in place of 10 inches; the enlarged scales of course admit of more detail in sub-division of the graduations and can be read more exactly, hence the accuracy is doubled. But to raise a three-figure to a five-figure accuracy the rule would have to be lengthened 100 fold—an eighty-foot rule is unwieldy.

A five-figure accuracy has been secured in two ways—first by Fuller, who arranges a logarithmic scale (41 ft. 8 in. long) helically on the outer surface of a hollow cylinder *I*. This cylinder corresponds to the movable disc in Boucher's Rule, it slides smoothly on a second hollow cyclinder *II*, which is twice as long as *I*, so as to allow *I* to be moved up or down through its own length as well as rotated upon it. The instrument is held by a handle attached to *II*. *II* also carries a fixed brass index *A* long enough to allow any number on *I* to be brought to its point. A movable index *B*, corresponding to the moving hand *B* in Boucher's Rule, is attached to a third cylinder *III* which slides within the hollow of *II*. The instrument is manipulated in exactly the same way as a Boucher. Thus for multiplication the graduated cylinder is slid round and down till its graduation "1" comes to the fixed index *A*. (This movement can be made rapidly by bringing a pin on the cylinder up to a fixed stop.) The movable index *B* now being brought to any number, say, 3216 on the scale, the distance from *A* to *B* measured along the helix represents the logarithm of 3216. Hence if the cylinder *I* be moved so that any other number, say 6158, comes to *A* the number which is now marked by *B* is log 3216 ahead of 6518, *i.e.* *B* marks 3216×6518.

The Fuller rules are beautifully made, the only drawback to them is that they have no *C* or *D* scale for square roots, and they do not give trigonometric ratios except by tables printed on the spare surface of the fixed cylinder *II*.

The other five-figure rule was designed by Thatcher. It is simply a 60-foot rule of the Gravet type; the scale *B* of the slide is cut up into 40 equal lengths and these are arranged along 40 equidistant generating lines of a 2-ft. cylinder, which corresponds to the slide of the straight rule. In Fig. 10, which is diagrammatic, *B* is shown cut into 12 lengths only; also for clearness the graduated strips are not shown in the right half of the figure. The scales *A* and *D* are also cut into 40 equal lengths, and these are arranged on the 40 faces of 20 V-shaped ribs* of a skeleton cylinder, which constitutes the "fixed" part of the rule and within which the first cylinder slides. Although this instrument is of American build it is gratifying to know that the plates from which the scales are printed were engraved by Messrs. Stanley of London. The instrument costs about £12, but the luxury of its use in five-figure work makes it well worth the outlay; it deals with x^2, x^3, $\sqrt{x}$, $\sqrt[3]{x}$, $x^{\frac{3}{2}}$, and $x^{\frac{2}{3}}$ as readily as does the Gravet, but it has no trigonometric scales.

The foregoing descriptions are of instruments applicable to calculations of every manner and kind; before the introduction of the "cursor," work involving several operations was tedious owing to the necessity of recording intermediate results, hence special rules were designed and adapted for the requirements of various trades or individuals; these are often very ingenious, they bear mysterious gauge points representing constant factors which enter

* Fig. 10 is drawn as though there were only 12 ribs, and they are cut away on the left so that only one of the supporting end-rings is seen.

into the formulae on which the results depend: the engineers' shibboleth $\cdot 7854\left(\frac{\pi}{4}\right)$, the number of cubic inches in a gallon (277), and the like, are samples of these mysteries. Sometimes these rules carry several slides so as to give the product of several factors simultaneously. There should be a tendency for them to become obsolete, but the interest of unearthing the mode of their construction must always to a mathematician be as intense as that afforded to an archaeologist by any cuneiform inscription.

An interesting artifice in these special rules is the use of a reversed scale. With two uniformly divided scales as in Fig. 11, if any two numbers (as 8 and 3) be brought together then as we pass along the scales the sum of the readings is obviously constant, and we can read the sum $8+3$ at the zero of either scale. So with two logarithmic scales as we pass along the scale we have a constant product, hence if 3 and 8 be brought together, as in Fig. 12, the product 8×3 can be read at the end of either scale.

In the very pretty H.P. rule designed by Hudson and engraved by Messrs. Stanley there are two slides with two pairs of reversed scales, one of them, on which cylinder diameters are read, is of double size so that the diameters are automatically squared. The scales effect the product of the four variable factors involved in "$\text{H.P.}=\frac{\text{p.l.a.n}}{33{,}000}$" and a suitable shifting of the zero on the final scale provides for the constant factors π, 33,000, etc.

A full description of this rule would be too lengthy, we therefore describe a rule designed on the same lines which will give at sight the volume and weight of a cylinder of known diameter, length, and density. In Fig. 13 the scale A is double the reversed scale B, hence along these scales instead of having a constant product of simple numbers, the number on A is squared in the product; thus we find $6^2\times 1$, $3^2\times 4$, $2^2\times 9$, $1\cdot 5^2\times 16$, $1^2\times 36$ all equal to 36, the number marked on B by the left end of A. The scales C and D are the same size as B, but the unity end of D has been set back to the left so that 7·854 on D comes over 1 on A. If the slide be removed from the rule, and we begin to replace it, the unity end of B first comes to the unity end of A, our constant product is $1^2\times 1=1$; the right end of C then marks 7·854 on D, which is the number of grammes in a cylinder diameter 1 cm., length 1 cm., and density 10. In the figure the slide has been pushed in from this position through a length representing log 36, therefore the right end of C now marks $7\cdot 854\times 36$ [$=283$] on D, while the unity indicated by a blurred arrow near the middle of C marks 28·3 on D; this is volume in c.c. of the cylinders, diameter 3 cm., length 4 cm., or diameter 2 cm., length 9 cm., etc. Finally, if we travel along the density line C from the arrow to say 7 the result on D is multiplied by 7, hence we read 198 grammes as the weight of each of these cylinders when the density is 7.

The following points should be looked to in selecting a slide rule:

(1) At each end of the rule a space of about $\frac{3}{4}''$ should be left clear beyond the graduations, so that the cursor may not wobble when used near the end of the scale: the writer and a colleague have for years attempted in vain to secure this improvement. Messrs. Faber have just issued a rule with 1·5 cm. clear at each end.

(2) The glass of the cursor should be quite flat on the rule and the index line should be very fine: any failing in these respects soon reduces the accuracy to $\frac{1}{2}\%$. It would be a great improvement if a break 2 mm. long were made in the line just where it crosses the top and bottom edge of the slide: in this way all readings would be made at the *end* of a fine line and the disturbing effect of thickness of line would be eliminated. Specimen cursors improved in this way have been promised to the writer by two English firms in succession, one promise dates three, the other six years back. M. Gravet of Paris has produced a specimen in three months, and

although the required improvement was not quite understood the writer hopes that at the next attempt he may attain his object.

In conclusion the writer would ask for indulgent criticism of the crude figures which illustrate this article; they were produced at a time when pressure of work prevented the use of the beam compass and proper dividing instruments, but it is hoped that they will serve their purpose as well as more artistic productions would have done. F. R. BARRELL.

REVIEWS.

The Theory of Equations, with an Introduction to the Theory of Binary Algebraic Forms. By W. SNOW BURNSIDE, M.A., D.Sc., and A. W. Panton, M.A., D.Sc. Fourth Edition. Vol. II., pp. xi. 292. (Dublin University Press, 1901).

Fourth editions do not often call for much reviewing. It, as a rule, suffices to give them a cordial welcome, and to congratulate the successful authors on still holding the field. There is no need to dwell again on the old merits which have long since secured recognition, and, as to old slips, such as there were have disappeared editions ago.

But the present notice must not be purely formal. Drs. Burnside and Panton do not rest content with stereotyping what has made its mark; they can always see room for improvement and expansion. The new edition appears in two volumes, and not like its predecessors in one only. This is a good thing. That the part which every mathematical schoolboy ought to possess, and the part which appeals to senior students should be procurable separately, is convenient for both classes of purchasers, and may not prove unprofitable to the publishers. Besides, the third edition was already too bulky for easy handling, and 82 additional pages have had to be found room for. The changes made in the old matter do not call for much remark. Suffice it to say that there is occasional rearrangement, and a fairly considerable addition to the number of worked examples which are of higher mathematical interest. Some additional space is devoted to semivariants, more particularly in a note which reproduces an investigation bearing on the question as to how many there are of a given type. The truly noteworthy addition to the book is, however, that of an entirely new last chapter of 50 pages, which marks quite a new departure in an English work on Equations.

This deals with the Theory of Substitutions and Groups of Substitutions, and its application. We have heard the slighting remark that English books on the 'Theory' of Equations do not justify their title, since the Resolvent 'Theory,' of which modern foreign treatises are full, has so far been absent from them. The stricture, no doubt far too sweeping, has yet been not without plausibility, so long as even the scientifically conceived, and, within its scope, thorough text-book of our present authors has only introduced reference to Abel and Galois once, and in a historical note. It may now be relegated to the well-grown category of taunts which are obsolete; for a new start has been made, and the one old defect in the admirable work which we are considering exists no longer.

Of course in but fifty pages of the bold type and ample spacing, for which poor eyes bless the press of the University of Dublin, there has been only room for a very brief treatment of so wide a theory as that of Substitutions and Resolvents. In the fifth edition we shall have more, no doubt. We are thankful, however, for what appears to be a very judicious selection from a great mass of material, and for an exposition which well exhibits the beauty of the Substitution-Group idea, and indicates convincingly the manner in which it has been brought to bear on deciding vexed questions of algebraical solubility. The subject is fascinating, but, as anyone who has tried to lecture on it will know, most difficult to grasp and to expound. Our authors are to be congratulated on their success in showing briefly its power as well as its attractiveness.

A final note of two pages deals with Abelian Equations. Let us hope for an ampler treatment of them in the next edition. E. B. E.

Leçons sur les Séries Divergentes. Par E. BOREL. Pp. viii. 184. 1901. (Paris: Gauthier-Villars.)

The author of the volume under review is well known to those who follow the progress of Pure Mathematics in France. His activity during the last few years has carried the modern analysis into a field of research which for eighty years has been almost completely ignored by mathematicians; for the study of Divergent Series was abandoned from the time of its overthrow by Cauchy* in 1821, until the work of Poincaré on Asymptotic Expansions and Borel on Summable Series in the last fifteen years.

The history of the use of Infinite Series is indeed one of the most interesting chapters in the book of mathematical evolution. First we have the great writers of the eighteenth century intoxicated by the power of the new engine of research, driving it right and left without scruple; Euler, for instance, gives the sum of the series

$$\ldots+\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\ldots$$

as zero, on the ground that

$$z+z^2+z^3+\ldots=\frac{z}{1-z},$$

and

$$1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots=\frac{z}{z-1}.$$

This age of recklessness was followed at the beginning of the nineteenth century by a movement of reaction, under the leadership of Gauss and Cauchy. The watchword of the new school was "Convergence"; series which are convergent may be used; series which are divergent must be rejected. And so a great part of the work of the previous generation was thrown on the rubbish-heap, there to lie until, in our own day, the French analysts have created of it a new science. The Englishman, practical rather than logical, and aware that the use of unjustifiable methods sometimes leads to correct results, has always secretly rebelled against the banishment of divergent series. He will be refreshed to hear that now the days of outlawry are over, and that series which do not converge may, under certain conditions, be applied in the same way as those which do.

Perhaps the best example of this advance in knowledge is furnished by the *Asymptotic Expansions* of M. Poincaré, the genesis of which we may now explain.

If we consider the differential equation

$$\frac{d^2u}{dz^2}+\left(\frac{3}{z}-\frac{1}{z^2}\right)\frac{du}{dz}+\frac{u}{z^2}=0,$$

we easily find that it is formally satisfied by the solution

$$u=-1-z-2z^2-6z^3-24z^4-\ldots-n!z^n-\ldots.$$

Unfortunately, however, this series is divergent for every value of z except zero; and consequently under the old rules, which excluded divergent series from all calculations, it would not have been possible to make any use of this information.

Let us, however, be reckless and perform some unjustifiable operations; we have on differentiating

$$\begin{aligned}\frac{d}{dz}\{e^{\frac{1}{z}}(-z-z^2-2z^3-6z^4-24z^5-\ldots-n!z^{n+1}-\ldots)\}&\\ =\frac{1}{z^2}e^{\frac{1}{z}}(z+z^2+2z^3+6z^4+24z^5+\ldots-)-e^{\frac{1}{z}}(1+2z+6z^2+24z^3+\ldots)&\\ =\frac{1}{z}e^{\frac{1}{z}},&\end{aligned}$$

the terms in the two series cancelling out. Thus

$$e^{\frac{1}{z}}(-z-z^2-2z^3-6z^4-24z^5-\ldots)=\int\frac{1}{z}e^{\frac{1}{z}}dz,$$

and so our series

$$-1-z-2z^2-6z^3-24z^4-\ldots$$

is transformed into

$$\frac{1}{z}e^{-\frac{1}{z}}\int_0^{\frac{1}{z}}\frac{e^v dv}{v}.$$

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* On the publication of Cauchy's *Analyse Algébrique*.

The series, as already said, is divergent, and therefore meaningless; but *this integral-expression is easily seen to be convergent and to actually represent a solution of the differential equation.* We have therefore, by a procedure hitherto regarded as illegitimate, obtained an accurate and valuable result. The applicability of divergent series in many such cases has long been known. The advance made by M. Poincaré is that we now know exactly *why* such a process gives a true solution, and so we are enabled to admit it into the ordinary machinery of orthodox analysis.

The divergent series above quoted is, in fact, what M. Poincaré calls an *asymptotic expansion.* The definition is, that a divergent series

$$a_0+a_1z+a_2z^2+a_3z^3+\ldots,$$

in which the sum of the first n terms may be denoted by s_n, is said to be an asymptotic expansion of a function $f(z)$, when the quantity

$$\frac{f(z)-s_n}{z^n}$$

tends to zero as z (supposed, say, to be real and positive) tends to zero. And M. Poincaré has proved that *asymptotic expansions may be added, multiplied, or integrated, just as if they were ordinary convergent series.*

This example will serve to illustrate the ideas of M. Borel's new volume. Into the several theories which occupy its chapters—asymptotic expansions, continued fractions and the theory of Stieltjes, M. Borel's own theory of summable series and its connection with the process of continuation, and the theory of expansions in series of polynomials—we cannot here enter in detail. But the book itself is short and well written, and we heartily commend it to the English mathematical world. E. T. WHITTAKER.

Non-Euclidean Geometry. By HENRY P. MANNING, Ph.D. (Ginn & Co., Boston, 1901; 8vo., pp. v.-95.)

It is interesting to note that in English appears the first text-book, the first class-book intended for elementary instruction in this fascinating subject, geometry non-Euclidean. It is destined to do a world of good in bringing home to teachers the scope and the meaning of geometry.

England has been so devoted to Euclid, that it may be noteworthy that the method of treatment taken as the basis of the first chapter, and which consequently underlies the other three chapters of this book, is from Saccheri's "*Euclid vindicated from every fleck.*" This has also a historic appropriateness, since modern researches in non-Euclidean geometry had their source in attempts, persisting through centuries but always failing, to prove superfluous the fifth postulate of Euclid, the so-called Parallel-postulate.

Was Saccheri's book simply one of those attempts? Did he really believe that his beautiful Non-Euclidean Geometry was nothing but a *reductio-ad-absurdum*? Did he really mean to justify the title of his book by proving non-Euclidean geometry untenable? May we not maintain that the great Jesuit was rather the forerunner of the new school, of the followers of Schweikart, Lobachevski, and Bolyai János, who believe that it is our non-Euclidean geometry itself which finally vindicates Euclid "from every fleck?" In order to guess Saccheri's own mind we have to remember, that in the close and constant intercourse with other mathematicians of his Order, his genius may have been intimidated by repeated discussions, and that before he could publish anything, the manuscript had to pass the censorship of at least two of the Jesuits. One can fully realize that he may not have dared to state, or may not have been allowed to publish the statement, that Euclid's geometry is only one of three possible systems. Yet his genius was leaning towards this statement.

That Manning has chosen Saccheri for his basis is further gratifying in view of the fact that in America and into English was made the first translation of Saccheri into a modern tongue. Manning takes as his equivalent for the parallel-postulate Euclid I. 32:—the three angles of a triangle equal two right angles. Accepting the infinity of the straight line, this equivalence amounts to the assumption of two theorems which it is convenient to call Legendre's theorems:

(1) In a triangle the sum of the three angles can never be greater than two right angles.

(2) If in any triangle the sum of the three angles is equal to two right angles, then so it is in every triangle.

The proof of these theorems by Saccheri, Legendre, and De Tilly is found, however, to assume what we call Archimedes' axiom, which on the other hand is not necessary in establishing Euclid's geometry. Thus these demonstrations did not prove the equivalence of Euc. I., 32 to Euc. Postulate 5. This extraordinarily interesting and important question has just been decided, and the outcome is a triumph of Euclid the eternal over Saccheri, Legendre, De Tilly, and all those who have built their geometries on such theorems as, "the angle-sum in a triangle is two right angles," "the equidistantial is a straight," etc., none of which can longer be considered the equivalent of Euclid's chosen postulate.

M. Dehn, in an investigation undertaken at the instigation of Prof. Hilbert (*Mathematische Annalen*, 53 Band, pp. 404-439), proves Legendre's second theorem without Archimedes' axiom or any assumption of continuity. This is a remarkable advance even over the proof given by Lobachevski and reproduced by Hilbert in his *Vorlesung ueber Euklidische Geometrie* (Wintersemester, 1898-99). But much more remarkable is M. Dehn's proof that the first Legendre theorem *never can be so proven.** He discusses the connection between the three different hypotheses relative to the sum of the angles and the three different hypotheses relative to parallels. He reaches the following surprising results: From the hypothesis that through a given point we can draw an infinity of parallels to a straight, if we exclude the axiom of Archimedes (given below), it *does not* follow that the sum of the angles of a triangle is less than two right angles, but on the contrary that this sum may be (*a*) greater than two right angles, (*b*) equal to two right angles. To demonstrate case (*a*) of this theorem, he constructs a geometry where we can draw through a point an infinity of parallels to a straight and where, besides, all the theorems of elliptic geometry are also verified. To this geometry he gives the name Non-Legendrean.

From the existence of this Non-Legendrean Geometry it follows immediately that it is impossible to demonstrate the first theorem of Legendre without using the axiom of Archimedes. To demonstrate case (*b*) he constructs a geometry without axiom of parallels, and where nevertheless are verified all the theorems of Euclidean geometry:—the sum of the angles of a triangle is equal to two right angles, similar triangles exist, the extremities of equal perpendiculars erected to a straight are all on one straight, etc. From the existence of this geometry it follows that, if we exclude the axiom of Archimedes, the parallel-postulate cannot be replaced by any of the propositions which have commonly been regarded as its equivalent. He gives this result as follows: These are Non-Archimedean Geometries, in which the parallel-axiom does not hold and yet the angle-sum in every triangle is equal to two right angles. Such a geometry he calls Semi-Euclidean. Finally he obtains the striking theorem: From the hypothesis of no parallel follows that the angle-sum is always greater than two right angles. So the two non-Euclidean hypotheses about parallels act in absolutely different ways in regard to the axiom of Archimedes.

We may exhibit the united results in the following table:

The angle-sum in a triangle is:	Through a point there is to a straight:		
	No parallel.	One parallel.	An infinity of parallels.
>2 R.	Elliptic Geometry.	(Impossible.)	Non-Legendrean Geometry.
$=2$ R.	(Impossible.)	Euclidean Geometry.	Semi-Euclidean Geometry.
<2 R.	(Impossible.)	(Impossible.)	Hyperbolic Geometry.

* This abstract, to the end of the table, is taken from pp. 109-110 of Laugel's translation of Hilbert, entitled *Les Principes Fondamentaux de la Géométrie* (Gauthier-Villars) 1900, and is not in the original German text. [W. J. G.]

From all this we see that Manning's Non-Euclidean Geometry is based on an axiom which he does not recognize, the axiom which Archimedes himself gives in the following form: If the point B lies on a straight between A and C and we construct the points B', B''... so that the order $ABB'B''$... holds, and further $AB = BB' = B'B''$..., then there is in the series of points B', B''... a point $B^{(n)}$ such that C lies between A and $B^{(n)}$. But since only one of Saccheri's three hypotheses needs this new assumption, it seems probable that the method adopted by Manning will not be repeated by subsequent authors. By laying down as universally valid the assumption that two straights can meet but once, Manning excludes spherical or double-elliptic geometry, which in fact he never even mentions, though Killing maintains that this was the only form of elliptic geometry which ever came before Riemann's mind. Russell even doubts any other form, saying (*Foundations of Geometry*, p. 39), "Klein has made great endeavours to enforce the distinction between the spherical and elliptic Geometries, but it is not immediately evident that the latter, as distinct from the former is valid." He also sums up his § 40 in the Table of Contents thus: "Klein's elliptic Geometry has not been proved to have a corresponding variety of space." But this, I take it, only shows that Russell even in 1897 did not know Killing's *Grundlagen der Geometrie* of 1893, which says, *e.g.*, p. 57, "Thereby we are led to two mutually exclusive possibilities, two forms of space. We will, in what follows, investigate both more precisely, and show that neither of them leads to a contradiction."

In Manning's Chapter I., III., 11, Theorem, p. 26, we need, to make his proof valid, a demonstration without parallels that the angle between the two perpendiculars at one point on the edge of a dihedral angle equals the angle between the two perpendiculars at any other point. This of course can be given by elementary geometry of congruence. The second chapter of 31 pages is taken from that part of Lobachevski which exists in English, namely his *Geometrical Researches on the Theory of Parallels*. No use is made of Bolyai's *Science Absolute of Space*, though that has also existed in English since June, 1891. Manning's third chapter, devoted to Elliptic Geometry, consists of only seven pages. Though it mentions that the straight is unilateral, it does not give the corresponding property of the elliptic plane. The final Chapter IV. is devoted to putting into coördinate and equational notation the new matter reached synthetically in the preceding chapters. The book ends with a Historical Note, pp. 91-95. In this note there are no serious errors. Manning's statement about Saccheri, though too brief, does not convey an incorrect or unfair impression.

I agree with the greatest living Jesuit mathematician, Professor John George Hagen, S.J., in thinking that Saccheri's elaborate and ingenious sequence of propositions was to develop non-Euclidean geometry in each case, and not to show that the systems were self-contradictory. Again when Manning says the Elliptic Geometry was discovered by Riemann he simply uses the term to include the single elliptic and double elliptic without going into the delicate historical question as to whether Riemann was conscious of both, a question which can never be decided.

Even in the authorized French translation (1900) of Hilbert's epoch-making *Grundlagen der Geometrie* the name *Géométrie de Riemann* is used for simple elliptic geometry.

On the whole, Manning's Non-Euclidean Geometry is highly creditable and is a permanent advance. GEORGE BRUCE HALSTED.

Austin, Texas.

A Treatise on Physics. By ANDREW GRAY, LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow. Vol. I. Dynamics and Properties of Matter. London: J. & A. Churchill, 1901. Pp. xxiii. 688.

This volume, the first instalment of a Treatise on Physics, is practically a handbook of Dynamics. The scheme of it is rather like that of the first volume of Thomson and Tait's *Natural Philosophy*; but it is a much less ambitious work, and, though it contains some original matter, is more of the nature of a sketch of the various parts of the subject as they appear in other books. The author has shown good judgment in his selection of what a student of Physics is likely to read and find useful. The Statics and Kinetics of particles, rigid bodies, and

fluids, Gravitation and Astronomical Dynamics, Tides, Elasticity, and Capillarity, are all treated with a light hand and in an interesting style, no topic being burdened with too much detail.

It would perhaps be too much to expect that a book which covers so wide a range should also be a model of accuracy, and as a matter of fact in this respect the standard attained is not a very high one. Setting aside points about which opinions may differ, a good many blemishes, such as trifling slips, false references, and misprints, may be found by anyone who turns over the pages in a cursory way; thus there can be little doubt that the total number which careful reading would disclose must be considerable.

The book begins with a hundred pages on Kinematics, and we then come to a discussion of the Laws of Motion. Dr. Gray's treatment of this topic, though suggestive, leaves the impression of being somewhat wanting in lucidity and in directness of reference to facts upon which the theory is based. The step which historically came first, namely, the identification of acceleration as the feature of motion which could conveniently be made the subject of laws, might have been more clearly put; and at the outset the reader is put a little off the right track by the observed motion of a falling body being referred to mutual action between it and the earth, without any reservation. Dr. Gray hampers himself by undertaking to discuss Newton's laws one by one, which it is not easy to do satisfactorily. The first law is introduced with the statement that it was Newton who first gave it, which cannot be justified historically unless in the rather forced sense that, though he used practically the same words as his predecessors, he may be supposed to have understood their meaning better. The question of axes of reference is not neglected, but after being raised several times, and dealt with to some extent correctly, is left rather obscure. In connection with the choice of a standard for measurement of time, it is suggested that a test of suitability may be found, but as to the character of such a test we have not sufficient guidance to enable us to say with certainty what the author means. It is not clear whether the law of motion is supposed to furnish a definition or only a practical measure; perhaps the weight of evidence is in favour of the former being the view held. As to the relativity of force we find some judicious remarks, but some vagueness as to the position in which the question is left. On the whole the author seems to be aiming in the right direction in most of what he says, and his account of the subject would probably have been clearer if it had been written with less dependence upon traditional formulas.

The chapter on General Dynamical Methods contains some passages to which exception may be taken. As to the generalized co-ordinates of a system, Dr. Gray does not begin by saying that the co-ordinates must be such that their values will determine the position of the system, at any rate at a given time; though this is implied in the equations used and added as a caution some pages later. He begins by saying that their independence does not justify the idea that the superposition in any order of a number of variations of them will lead to the same resultant displacement. Thus the turning of a rigid body through successive angles α, β, γ, about the axes of x, y, z, will lead to different results according to the order in which the rotations are taken. It is difficult to see any reason for the mention of α, β, γ, unless for the purpose of warning the reader that, as their values do not determine the position of the body even within a limited region, they will not serve as co-ordinates. As the statement stands it is a warning that order of variation must be attended to, which is not the case. The whole subject of co-ordinates, which it is important to make clear, is treated in a confusing way. We are told that the generalized co-ordinates need not represent merely position co-ordinates, but may be any quantities characteristic of the motion of the system. This is not true, though it may be suggestive. We finally get to a more correct statement, which is followed by the remark that neglect of this point has led some writers into very serious error, such as the use of the angular velocities of a rigid body about its principal axes as if they were fluxions of co-ordinates. The case of these unfortunate writers, whoever they are, seems hardly worth raking up. The first practical example dealt with in this chapter is that of a top. The angular velocity of a top about its axis of symmetry OC "*fixed in the body*" is called ω. It appears, however, that this is not to be the angular velocity in the usual sense, but is to be that which is zero if the top always turns the same face to the vertical. This secures that ω is a fluxion of a co-ordinate, a fact which it might

have been well to mention explicitly at the outset as the object of this choice of notation. Then we are introduced to the angular velocity in the ordinary sense, and this is called "the angular velocity about an axis *fixed in space* and coinciding with the instantaneous position *OC*." The words used cannot be construed so as to express with any correctness the distinction between these two angular velocities; they are mere jargon of a confusing kind.

The short chapter on Graphical Statics is not a very satisfactory one. For parallel forces the case of any number of forces would be nearly as simple as that of only two, and have the advantage of illustrating the value of the method. Reciprocal figures are mentioned, but for the construction of diagrams the method afforded by reciprocity, which seems the obvious one to adopt, is not used. The statement is made that, in order that a closed frame in which the bars meet in n points may be stiff, it is necessary and sufficient that there should be $2n-3$ connecting bars. This however is not true.

In the account given of the theory of elasticity, the condition in the theory of small strain, that the rotation of any portion of the body relative to any other portion must be small, might have been made more clear. In the investigation of the stress strain equations for an isotropic body, the author begins with the relations between principal stresses and principal strains. It is, as a matter of fact, nearly as simple to find the general equations by superposition at once. The work of finding these equations is not given in detail, we are only told how by a process of transformation they may be found; and as they can be written down without any such process a false impression is likely to be conveyed. A brief account being given of St. Venant's theory of torsion, it seems a pity not to add the few lines which would have been sufficient to explain the idea upon which St. Venant's work is based, as this has an important bearing upon the applications of it. The important numerical results which St. Venant obtained for torsional rigidities might conveniently have been mentioned. We find an inconsistency in the use of the term elastic limit, in one place defined by the failure of Hooke's law, and in another place by the occurrence of set. We are told that the discussion for æolotropic bodies is very much more complicated than for isotropic bodies; which is true enough in general, but is too sweeping a statement in the face of the exceptions. St. Venant worked out his theory of prisms with three elastic constants, and showed that this made no additional complication.

The book contains a great deal of information, and is interesting, and seems likely to be useful; but it is greatly in need of further revision.

W. H. MACAULAY.

MATHEMATICAL NOTES.

101. [L¹. 7. a.] *To find the foci of the conic inscribed in the triangle of reference.*

The straight line $\Sigma l\alpha=0$ touches the conic $\Sigma\sqrt{\lambda\alpha}=0$ if $\Sigma\lambda/l=0$.

If $(\alpha_1, \beta_1, \gamma_1)$ is a focus, and ρ the semi-axis of the conic, the other focus is $\rho^2/\alpha_1, \rho^2/\beta_1, \rho^2/\gamma_1$.

The product of the perpendiculars from the foci on any tangent is constant $(=\rho^2)$.

Hence $$[\Sigma l\alpha_1 . \Sigma l\rho^2/\alpha_1 + P\Sigma\lambda mn]/(\Sigma\lambda^2 - 2\Sigma mn\cos A)=\rho^2;$$

$$\therefore\ \rho^2\{\beta_1/\gamma_1+\gamma_1/\beta_1\}+P\lambda=-2\rho^2\cos A$$

with two similar equations;

$$\therefore \text{ foci are given by } \alpha(\beta^2+\gamma^2+2\beta\gamma\cos A) : \ldots : \ldots = \lambda : \mu : \nu.$$

Any two cubics intersect in A, B, C, the two focoids and the four foci of the conic.

R. F. DAVIS.

102. [K. 11. b.] *On the circles of similitude relating to the in- and ex-circles of a triangle.*

1. It is an elementary proposition that the three circles of similitude of any three circles taken in pairs are coaxal. It is further easily seen that the

common axis always passes through the circumcentre of the three centres of the original circles, and also through the radical centre of the three circles.

Let the equations to the three circles be $S_1 \equiv (x-a_1)^2+(y-b_1)^2-r_1^2=0$, $S_2=0$, and $S_3=0$.

Then the equations to the three circles of similitude are $r_2^2S_1-r_1^2 . S_2=0$, $r_3^2S_1-r_1^2S_3=0$, and $r_3^2S_2-r_2^2S_3=0$.

The equation to the common radical axis of these three similitude circles is

$$\frac{r_2^2S_1-r_1^2S_2}{r_2^2-r_1^2}=\frac{r_3^2S_1-r_1^2S_3}{r_3^2-r_1^2}.$$

If P denotes any point on this axis and A, B, C be the centres of the three original circles, the above equation is the locus of P such that

$$\frac{r_2^2 . PA^2-r_1^2 . PB^2}{r_2^2-r_1^2}=\frac{r_3^2 : PA^2-r_1^2 . PC^2}{r_3^2-r_1^2},$$

i.e.

$$\Sigma PA^2(r_2^2-r_3^2)=0.$$

This relation is clearly satisfied at the circumcentre where $PA=PB=PC$, and also at the radical centre where $PA^2-r_1^2=PB^2-r_2^2=PC^2-r_3^2$.

It may be noticed that the three circles of similitude will in general intersect in two points. Hence it is in general possible to find two points in a plane such that any three circles in that plane will subtend the same angle at each point. Applying similar considerations to three spheres we see that if an eye is located anywhere on the circle which is the common intersection of the three spheres of similitude, the three spheres will all appear of the same size.

2. *If any set of three circles be chosen from the inscribed and escribed circles of any triangle, the three similitude circles of these taken in pairs have a common radical axis which passes through the orthocentre of the triangle.*

Using trilinears and taking the triangle as the triangle of reference, the equations to the in- and ex-circles are

$$K=\Sigma a^2(s-a)^2\alpha^2-2\Sigma bc(s-b)(s-c)\beta\gamma=0;$$

$$L=a^2s^2\alpha^2+b^2(s-c)^2\beta^2+c^2(s-b)^2\gamma^2-2bc(s-b)(s-c)\beta\gamma+2cas(s-b)\gamma\alpha+2abs(s-c)\alpha\beta=0.$$

$$M=\ldots \text{etc.}=0;$$

$$N=\ldots \text{etc.}=0.$$

These equations are such that on substituting the coordinates of any point in them the result $=m\times$ sq. of tangt. from point, where $m=4S^2$, as is found by substituting the coordinates of one of the angular points of the triangle.

The radii of the circles are $\frac{S}{s}, \frac{S}{s-a}, \frac{S}{s-b}, \frac{S}{s-c}$.

Choosing the first three circles the corresponding circles of similitude are

$$s^2 . K=(s-a)^2 . L,$$

$$s^2 . K=(s-b)^2 . M,$$

$$(s-a)^2 . L=(s-b)^2 . M.$$

Writing the first two of these equations out and simplifying, they become

$$\beta^2\cos B+\gamma^2\cos C-(\cos B+\cos C)\beta\gamma-(1+\cos A)\gamma\alpha-(1+\cos A)\alpha\beta=0,$$

and

$$\alpha^2\cos A+\beta^2\cos B-(1+\cos C)\beta\gamma-(1+\cos C)\alpha\gamma-(\cos A+\cos B)\alpha\beta=0,$$

which may be written

$$-\frac{b+c}{bc}(\Sigma a\beta\gamma)+(\Sigma a\alpha)\left(\frac{\cos B}{b}.\beta+\frac{\cos C}{c}\gamma\right)=0,$$

and

$$-\frac{a+c}{ac}(\Sigma a\beta\gamma)+(\Sigma a\alpha)\left(\frac{\cos A}{a}.\alpha+\frac{\cos C}{c}\gamma\right)=0.$$

The radical axis of these two circles is

$$\frac{c}{b+c}\cdot\cos B\,.\,\beta+\frac{b}{b+c}\cdot\cos C\,.\,\gamma=\frac{c}{a+c}\cdot\cos A\,.\,\alpha+\frac{a}{a+c}\cdot\cos C\,.\,\gamma;$$

i.e.
$$(b+c)\cos A\,.\,\alpha-(a+c)\cos B\,.\,\beta+(a-b)\cos C\,.\,\gamma. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

(1) is thus the equation to the radical axis common to the three circles of similitude belonging to K, L, and M.

Similarly choosing all possible groups of three from the four circles K, L, M, N, we find (2), (3), and (4) as the equations of the radical axes common to each set of corresponding circles of similitude,

$$(b-c)\cos A\,.\,\alpha+(a+c)\cos B\,.\,\beta-(a+b)\cos C\,.\,\gamma=0, \quad \ldots\ldots\ldots\ldots(2)$$
$$-(b+c)\cos A\,.\,\alpha+(c-a)\cos B\,.\,\beta+(a+b)\cos C\,.\,\gamma=0, \quad \ldots\ldots\ldots\ldots(3)$$
$$(b-c)\cos A\,.\,\alpha+(c-a)\cos B\,.\,\beta+(a-b)\cos C\,.\,\gamma=0. \quad \ldots\ldots\ldots\ldots(4)$$

(1), (2), (3), and (4) are all satisfied by $\alpha\cos A=\beta\cos B=\gamma\cos C$, and consequently all pass through the orthocentre.

Further, each passes through the radical centre of the three corresponding original circles and through the circumcentre of their centres.

The University of Adelaide, South Australia. R. W. Chapman, M.A.

Mr. R. F. Davis adds the following notes:

(1) The vertex and the point L, where the internal bisector of A meets the base BC, are respectively the ex- and in-centres of similitude of the in- and ex-circle opposite A.

(2) Hence the circle of similitude is the circle on AL as diameter. It passes through X, the foot of the perpendicular from A on BC. Expressing the condition that the lines $\beta/\beta_1-\gamma/\gamma_1=0$ and $(\beta-\gamma)/(\beta_1-\gamma_1)=\alpha/\alpha_1$ are at right angles, its trilinear equation is

$$(\beta_1-\gamma_1)(\beta_1\cos B-\gamma_1\cos C)=(1+\cos A)\alpha_1(\beta_1+\gamma_1).$$

(3) The three circles described on the internal bisectors are coaxal. [A geometrical proof is invited.]

(4) The orthocentre is the centre of similitude of these three circles, for

$$AH\,.\,HX=BH\,.\,HY=CH\,.\,HZ.$$

(5) If AP, BQ, CR be any lines from the vertices to the opposite sides, the radical centre of the circles on these lines as diameters is the orthocentre. [M'Clelland, 12, p. 181.]

PROBLEMS.

[*Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by their Proposers.*]

440. [L'. 14. a.] If e, E be two concentric coaxal ellipses, the lengths of their axes being respectively $(2a, 2b)$ and $(2A, 2B)$, there is an infinity of triangles inscribed to E and circumscribed to e, provided

$$\frac{a}{A}+\frac{b}{B}=1.$$

(1) The normals to E at the vertices of any one of these triangles PQR are concurrent, and so are the normals to e at its points of contact with the sides of PQR.

(2) Find the locus of each of these points of contact.

(3) Find the envelope of the chords SP, SQ, SR, where S is the fourth point of intersection of E with the circumcircle of PQR.

(4) Calculate the power of the common centre of the ellipses with respect to circle PQR.

(5) Find the loci of the orthocentre, the circumcentre, and the centroid of the triangle PQR.

(6) Calculate the area, the sum of the squares of the sides, and the sum of the squares of the sines of the angles of the triangle PQR.

Examine the following particular cases:

(i) $A=2a$; $B=2b$.

(ii) $A=B=a+b$.

(iii) $a=b=\dfrac{AB}{A+B}$.

(iv) $Aa=Bb=a^2+b^2$.

(v) $A=\dfrac{a^2}{a-b}$; $B=a$.

E. N. BARISIEN.

441. [K. 14. b.] If α, β, γ, δ are four points in the faces of a tetrahedron $ABCD$, from which the lengths of the perpendiculars on the sides of the triangles in which they lie are functions (and all are the same functions—except for a factor which may be different for the different faces) of the lengths of these sides, and if $A\alpha$, $B\beta$ are concurrent, so too are $C\gamma$, $D\delta$.

The incentre and symmedian point are particular cases: show that it can be extended also to the excentres. C. V. DURELL.

442. [L'. 1. b.] [Mechanical Construction of a Hyperbola.] A triangle slides with its base on a fixed straight line: prove that the line joining a fixed point to a carried point cuts a side in a point whose locus is a hyperbola; and find the asymptotes. R. W. H. T. HUDSON.

443. [J. 1. a.] A pack of mn cards is dealt out card by card into n heaps; the heaps are taken up in order and the pack again dealt out into n heaps; and so on. After how many deals is the original order (a) restored, (b) reversed?

If the second dealing be into m heaps the original order is reversed or restored according as the first heap in each dealing is top or bottom of the new pack. C. S. JACKSON.

444. [K. 2. e.] A line drawn arbitrarily through the centroid of an equilateral triangle ABC cuts the sides in A', B', C': show that the circles on AA', BB', CC' as diameters are coaxal and that their common points lie one on the incircle and the other on the circumcircle of ABC. V. RETALI.

445. [L'. 14. a.] The equation to the Pascal line of a hexagon inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$\begin{vmatrix} x/a. & y/b. & 1. & 0. \\ \cos\tfrac{1}{2}(\theta_1+\theta_4), & \sin\tfrac{1}{2}(\theta_1+\theta_4), & \cos\tfrac{1}{2}(\theta_1-\theta_4), & \sin\tfrac{1}{2}(\theta_1-\theta_4) \\ \cos\tfrac{1}{2}(\theta_3+\theta_6), & \sin\tfrac{1}{2}(\theta_3+\theta_6), & \cos\tfrac{1}{2}(\theta_3-\theta_6), & \sin\tfrac{1}{2}(\theta_3-\theta_6) \\ \cos\tfrac{1}{2}(\theta_5+\theta_2), & \sin\tfrac{1}{2}(\theta_5+\theta_2), & \cos\tfrac{1}{2}(\theta_5-\theta_2), & \sin\tfrac{1}{2}(\theta_5-\theta_2) \end{vmatrix} = 0$$

where θ_1, etc., are the eccentric angles of the vertices taken in order.

D. O. S.

446. [K. 21. a. β.] Construct a regular octagon by the use of the compasses only. ISAAC H. TURRELL.

447. [R. 7. b. γ.] A shot of mass m is fired from a gun of mass M which is movable on a smooth horizontal plane; if the muzzle velocity of the shot relative to the gun is given, and the position of the muzzle when the shot leaves is always the same, find the envelope of trajectories.

ST. JOHN'S (C.), 1895.

448. [R. 1.] A cycloid has its base vertical: find the inclination to the horizon of the time of quickest descent from a given point to the curve. C.

449. [K. 20. d.] If θ, ϕ are very small, show that

$$\frac{\theta}{\phi}=\frac{2}{3}\frac{\sin\theta}{\sin\phi}+\frac{1}{3}\frac{\tan\theta}{\tan\phi}-\frac{\theta}{180\phi}(\theta^2-\phi^2)(9\theta^2-\phi^2).$$ C.

SOLUTIONS.

UNSOLVED QUESTIONS.—171, 275, 279, 283, 326, 336-8, 341, 349, 356, 370, 373-4, 376-9, 381-2, 389, 393, 395-7, 401, 404, 406, 414, 421-3, 425-9.

Solutions to the above, or other questions to which no solution has yet been published, and to 430-49 should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be **very carefully drawn to a small scale** on a separate sheet.

102. [K. 20. e. a.] *Prove that the sum of the squares of the reciprocals of the lines joining one vertex of a regular polygon of n sides to the other $n-1$ vertices is $\frac{n^2-1}{12R^2}$, where R is the radius of the circumscribing circle.* Queens' (C.), '96.

Solution.

$$\text{Sum}=\frac{1}{4R^2}\sum_{r=1}^{n}\operatorname{cosec}^2\frac{r\pi}{n}$$

$$=\frac{n^2-1}{3}\cdot\frac{1}{4R^2}.$$

(Loney, Part II., 1894, p. 73, No. 9.)

106. [L¹. 2. c.] *Chords of a conic are drawn through a fixed point, and through the pole of each chord a straight line is drawn parallel to the chord; show that the lines so drawn touch a parabola.* Queens' (C.), '96.

Solution.

Let $ax^2+by^2=1$ be the conic and α, β the fixed point.

If $l(x-\alpha)+m(y-\beta)=0$ and $axx'+byy'-1=0$ be the same line, we have

$$\frac{l}{ax'}=\frac{m}{by'}=l\alpha+m\beta. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

A parallel through $x'y'$ to the chord is

$$l(x-x')+m(y-y')=0,$$

or $$l^2(abax-b)+ablm(\beta x+\alpha y)+m^2(ab\beta y-a)=0 \text{ [from (1)]};$$

$\therefore$ envelope is $$ab(\beta x+\alpha y)^2=4(a\alpha x-1)(b\beta y-1),$$

or $$ab(\beta x-\alpha y)^2=4(1-a\alpha x-\beta by), \text{ a parabola.}$$

108. [L¹. 10. b.] *Find the equation to the (Joachimsthal's) circle through the feet of the normals from any point (x', y') to a parabola.* Queens' (C.), '96.

v. Casey's *Conics* (1893), p. 185.

109. [L¹. 7. d.] *If P be any point on an ellipse, foci S, S', semi-axes a, b, and PS, PS' meet the ellipse in Q, Q', show that the envelope of QQ' is an ellipse of semi-axes a, $b^3/(2a^2-b^2)$.* Clare and Trin. Hall, '96.

Solution.

Let the eccentric angles of P, Q, Q' be α, β, γ.

Then $$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=-\frac{1+e}{1-e}, \quad \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}=-\frac{1-e}{1+e},$$

$$QQ' \text{ is } \frac{x}{a}\cos\frac{\beta+\gamma}{2}+\frac{y}{b}\sin\frac{\beta+\gamma}{2}=\cos\frac{\beta-\gamma}{2},$$

or on substituting and arranging,

$$\cot^2\frac{\alpha}{2}\left(1+\frac{x}{a}\right)+2\cot\frac{\alpha}{2}\cdot\frac{y}{b}\left(\frac{1+e^2}{1-e^2}\right)+1-\frac{x}{a}=0;$$

$\therefore$ envelope is $$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(\frac{1+e^2}{1-e^2}\right)^2=1,$$

an ellipse with the given semi-axes.

110. [L¹. 5. a.] *Find the equation of the normal at the point $\theta=\alpha$ of the conic $l=r(1+e\cos\theta)$, and prove the part intercepted by the curve subtends at the origin an angle* $2\tan^{-1}\dfrac{1+e^2+2e\cos\alpha}{e\sin\alpha}$. Gonv. and Caius, 1895.

Solution.

The normal $$\frac{le\sin\alpha}{1+e\cos\alpha}\cdot\frac{1}{r}=e\sin\theta+\sin(\theta-\alpha)$$

cuts curve where $$l=r(1+e\cos\theta),$$

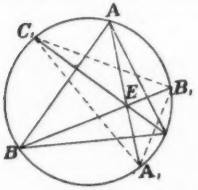
i.e. where $$\tan\frac{\theta}{2}=\frac{e+P}{e-P}\cot\frac{\alpha}{2}\quad(P=1+e\cos\alpha+e^2).$$

The angle subtended $=\theta-\alpha=2\phi$ (say)

$$\tan\phi=\tan\frac{\theta-\alpha}{2}=\text{(on reduction)}\quad\frac{P+e\cos\alpha}{e}\operatorname{cosec}\alpha=\frac{1+2e\cos\alpha+e^2}{2e\sin\alpha}.$$

$$\therefore\ \phi=2\tan^{-1}\frac{1+e^2+2e\cos\alpha}{2e\sin\alpha},\text{ and not as stated.}$$

364. [K. 5. a.] *Points E, E' are taken in the triangles ABC, $A'B'C'$ respectively, such that $B\hat{E}C=B'\hat{E}'C'=A+A'$; $C\hat{E}A=C'\hat{E}'A'=B+B'$. Show that the lines EA, EB, EC; $E'A'$, $E'B'$, $E'C'$ divide the two triangles into mutually similar pairs of triangles.* R. F. Muirhead.



Solution by C. E. Youngman.

Draw the circle ABC cutting AE, BE, CE at A_1, B_1, C_1. Then $\angle B_1A_1C_1=B_1BC_1=BEC-BCC=A'$; and similarly $B_1=B'$, $C_1=C'$. Also $\angle B_1EC_1=BEC=B'E'C'$, and so on; $\therefore$ the figure $A_1B_1C_1E$ is similar to $A'B'C'E'$. But ECB is similar to EB_1C_1; $\therefore$ etc.

375. [K. 20. e.] *A famous stiltwalker bets that he will walk into the sea at low tide on a pair of stilts and remain there with his feet unwetted until high tide. Learning that he has miscalculated the height of the tide, he consults a mathematical friend, who measures a base of 50 feet on the sand, and then observes the angles, which the lowest point to which the tide falls on a perpendicular inaccessible rock makes with the ends of this base, to be 63° 4′ and 58° 2′ respectively. At the same time he observes the angles made with the same points by the highest perpendicular point at which the seaweed grows to be 63° 20′ and 58° 16′ respectively, and also the angle subtended at the first end of the base by the two points of rock to be 23° 18′. Find the height of the tide.* T. Roach.

Solution by Proposer.

Let AB be the base; P, Q the two points of rock,

$$PAB=63°\,4',\quad PBA=58°\,2',\quad AB=50,$$

$$\log PA=\log 50+L\sin 58°\,2'-L\sin 58°\,54'=1{\cdot}6989700+9{\cdot}9285783-9{\cdot}9326092$$
$$=1{\cdot}6949391=\log 49{\cdot}538,$$

$$QAB=63°\,20',\quad QBA=58°\,16',$$

$$\log QA=\log 50+L\sin 58°\,16'-L\sin 58°\,24'=1{\cdot}6989700+9{\cdot}9296770-9{\cdot}9303004$$
$$=1{\cdot}6983466=\log 49{\cdot}928,$$

$$QAP=23°\,18',$$

$$L\tan\frac{APQ-AQP}{2} = \log\overline{AQ-AP} - \log\overline{AQ+AP} + L\cot 11°39'$$
$$= \log\cdot 49 - \log 99\cdot 466 + L\cot 11°39'$$
$$= \bar{1}\cdot 6901961 - 2\cdot 9976747 + 10\cdot 6857532$$
$$= 8\cdot 3782746 = L\tan 1°22'8'',$$

$$\left.\begin{aligned}\frac{P-Q}{2} &= 1°22'8''\\ \frac{P+Q}{2} &= 78°21'\end{aligned}\right\} \begin{aligned}P &= 79°43'8''\\ Q &= 76°58'52''\end{aligned}$$

$$\log QP = \log AP + L\sin 23°18' - L\sin 76°58'52''$$
$$= 1\cdot 6949391 + 9\cdot 5971965 - 9\cdot 9886908$$
$$= 1\cdot 3034448 = \log 20\cdot 112,$$

$$\therefore\ QP = 20\cdot 112.$$

380. [L¹. 11. c.] (1) *A chord of a rectangular hyperbola subtends a right angle at a fixed point. Find geometrically the envelope of the chord.*
(2) *A and B move on axes Ox, Oy, so that the perimeter of the triangle OAB remains constant. Find the envelope of the circle OAB.* E. N. BARISIEN.

Solution by C. E. YOUNGMAN.

(1) Let S be the fixed point, AB one of the chords, and C the point at which SA meets the hyperbola again. Then H the orthocentre of ABC lies on the hyperbola; complete the quadrilateral $ABCH$—the third diagonal DE must be a fixed line, being the polar of S. Draw STM perpendicular to AB to meet AB at T and DE at M. Then, because DE and SE are equally inclined to AB, $ST=TM$; $\therefore$ T moves on a straight line half-way between S and DE, that is, the pedal for S of the envelope of AB is a straight line. Hence AB touches the parabola whose focus is S, and directrix DE.

(2) One of the circles which can be drawn to touch OA, OB, and the circle OAB, is such that the polar of O passes through I' the excentre opposite to O (see vol. I., p. 190). But I' is fixed, because the projection of OI' on OA or OB is half the constant perimeter; hence the circle is fixed, and must be the envelope required.

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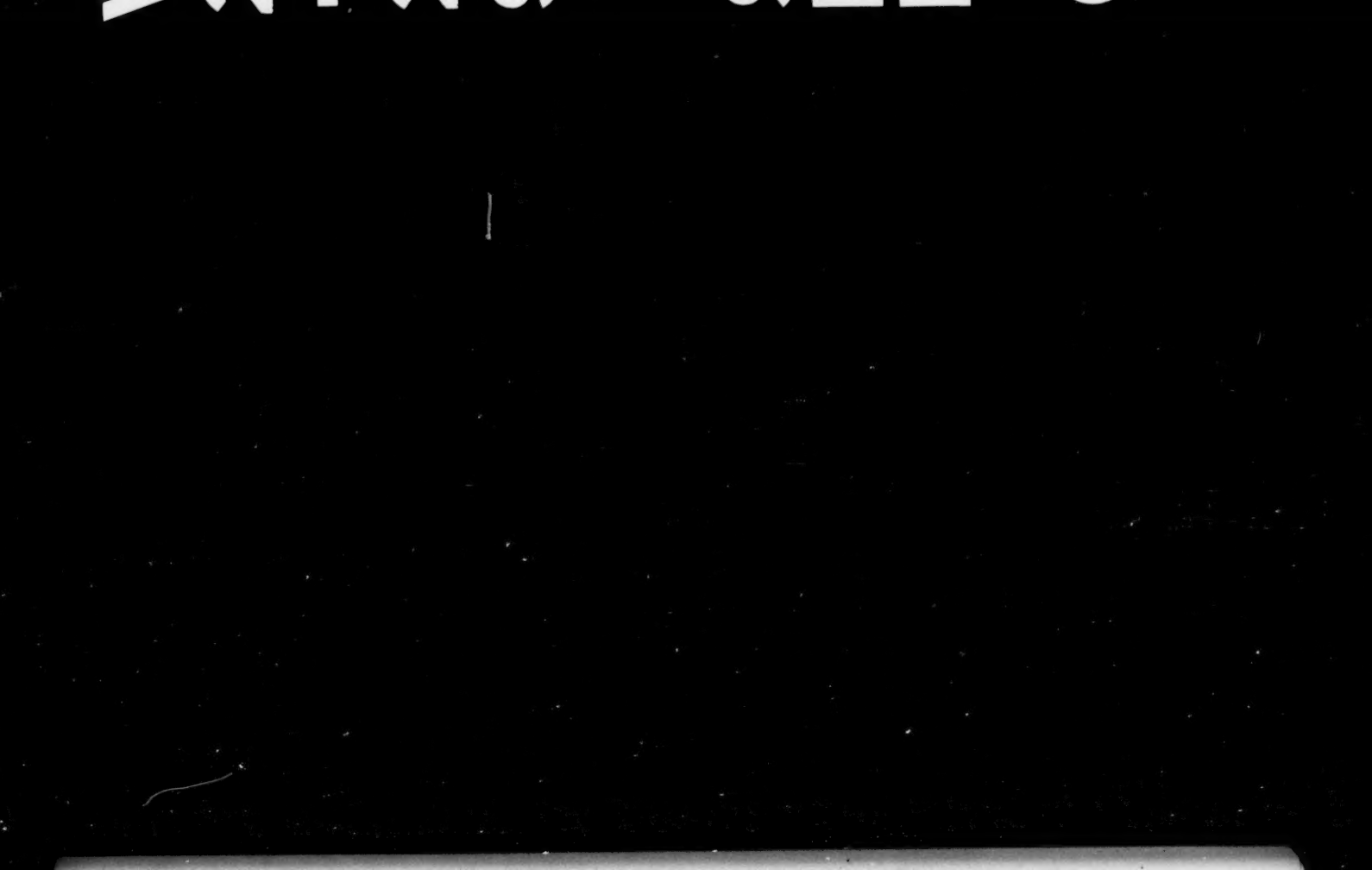
* Will be shortly reviewed.

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